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# Critical properties of the Calogero–Sutherland model with boundaries

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**Abstract.** Critical properties of the Calogero–Sutherland model of  $BC_N$ -type ( $BC_N$ -CS model) are studied. Using the asymptotic Bethe ansatz spectrum of the  $BC_N$ -CS model, we calculate finite-size corrections in the energy spectrum. Since the  $BC_N$ -CS model does not possess translational invariance, the finite-size spectrum acquires the contributions coming from ‘boundaries’. We show that the low-energy critical behaviour of the model is described by  $c = 1$  boundary conformal field theory. Thus the universality class of the model is identified as a chiral Tomonaga–Luttinger liquid.

## 1. Introduction

The Calogero–Sutherland (CS) models [1–3] describe one-dimensional quantum many-body systems with inverse-square long-range interactions. Among many variants of the CS model [4], a class of models which are not translationally invariant has been found over the past years [5]. In particular the so-called CS model of  $BC_N$ -type (abbreviated as the  $BC_N$ -CS model hereafter) is the most general model with  $N$  interacting particles. The  $BC_N$ -CS model is intimately related to the root system of type  $BC_N$  and invariant under the action of the Weyl group of type  $B_N$ . Namely, the model is invariant under coordinate transformations

$$(q_1, q_2, \dots, q_N) \mapsto (\epsilon_1 q_{\sigma(1)}, \epsilon_2 q_{\sigma(2)}, \dots, \epsilon_N q_{\sigma(N)}) \quad (1)$$

where  $(q_1, q_2, \dots, q_N) \in \mathbb{R}^N$  denote the coordinates of  $N$  particles,  $\epsilon_j \in \{\pm 1\}$  and  $\sigma$  is an element of the symmetric group of  $N$  letters. Roughly speaking, the Weyl group of type  $B_N$  consists of the ordinary exchange of particle coordinates and the sign change of coordinates. As we will see below the latter is understood as the mirror image of particles with respect to a boundary.

Recent works have made it clear that the  $BC_N$ -CS model is relevant to one-dimensional physics with boundaries. For instance, it was pointed out that the non-relativistic dynamics of quantum sine–Gordon solitons in the presence of a boundary is described by the  $BC_N$ -CS model (with sinh interaction) [6]. This model is interesting in view of the quantum electric transport in mesoscopic systems [7, 8]. The Haldane–Shastry model, which is the discrete version of the CS model, with open boundary conditions can also be constructed by utilizing the root system of type  $BC_N$  [9, 10]. We shall present further evidence for the relevance of the  $BC_N$ -CS model to our understanding of one-dimensional physics including boundary effects.

In this paper we will analyse the long-distance critical properties of the  $BC_N$ -CS model. Since the exact energy spectrum of the model is available [11], we may apply the method of finite-size scaling developed in conformal field theory (CFT) to study the critical behaviour. The same technique has already been employed when the critical properties of the CS model of  $A_{N-1}$ -type were considered [12]. The universality class of the  $A_{N-1}$ -CS model is identified as a Tomonaga–Luttinger liquid which is equivalent to  $c = 1$  Gaussian CFT. In what follows we will show that, in contrast to the  $A_{N-1}$ -CS model, the  $BC_N$ -CS model exhibits the critical behaviour described by  $c = 1$  CFT with boundaries [13]. Hence the universality class will be found to be a chiral Tomonaga–Luttinger liquid [14].

In the next section we first introduce the  $BC_N$ -CS model and review the energy spectrum of the model obtained by using the asymptotic Bethe ansatz. In section 3 we consider the thermodynamic properties. In section 4 the finite-size scaling analysis of the energy spectrum is performed. Finally, in section 5, we discuss various critical exponents of correlation functions.

## 2. The $BC_N$ -CS model

Let us write down the Hamiltonian of the  $BC_N$ -CS model [5]. We put the system in finite geometry with linear size  $L$  and impose periodic boundary conditions. The Hamiltonian is then given by

$$\begin{aligned} \mathcal{H} = & - \sum_{j=1}^N \frac{\partial^2}{\partial q_j^2} + 2\lambda(\lambda - 1) \left(\frac{\pi}{L}\right)^2 \sum_{1 \leq j < k \leq N} \left\{ \frac{1}{\sin^2 \frac{\pi}{L}(q_j - q_k)} + \frac{1}{\sin^2 \frac{\pi}{L}(q_j + q_k)} \right\} \\ & + \lambda_1(\lambda_1 + 2\lambda_2 - 1) \left(\frac{\pi}{L}\right)^2 \sum_{j=1}^N \frac{1}{\sin^2 \frac{\pi}{L} q_j} + 4\lambda_2(\lambda_2 - 1) \left(\frac{\pi}{L}\right)^2 \sum_{j=1}^N \frac{1}{\sin^2 \frac{\pi}{L} 2q_j} \end{aligned} \quad (2)$$

where  $\lambda$ ,  $\lambda_1$  and  $\lambda_2$  are coupling constants which are assumed to be non-negative. It is clearly seen that the Hamiltonian (2) is invariant under the action (1) of the Weyl group of type  $B_N$ . There exist several interaction terms which will need explanation. The term  $1/\sin^2(\pi/L)(q_j + q_k)$  expresses the two-body interaction between the  $j$ th particle and the ‘mirror image’ (we place a mirror at the origin  $q = 0$ ) of the  $k$ th particle ( $j \neq k$ ). The term  $1/\sin^2(\pi/L)q_j^2$  may be interpreted as the potential due to *impurity* located at the origin. The term  $1/\sin^2(\pi/L)2q_j$  describes the interaction between the  $j$ th particle and its own ‘mirror image’. All these terms required by invariance under the action of the Weyl group of type  $B_N$  violate translational invariance. Therefore, the total momentum is not a good quantum number for the  $BC_N$ -CS model.

The Hamiltonian (2) can be cast into another form just by using the elementary identity  $\sin 2A = 2 \sin A \cos A$ . One obtains

$$\begin{aligned} \mathcal{H} = & - \sum_{j=1}^N \frac{\partial^2}{\partial q_j^2} + 2\lambda(\lambda - 1) \left(\frac{\pi}{L}\right)^2 \sum_{1 \leq j < k \leq N} \left\{ \frac{1}{\sin^2 \frac{\pi}{L}(q_j - q_k)} + \frac{1}{\sin^2 \frac{\pi}{L}(q_j + q_k)} \right\} \\ & + \mu(\mu - 1) \left(\frac{\pi}{L}\right)^2 \sum_{j=1}^N \frac{1}{\sin^2 \frac{\pi}{L} q_j} + \nu(\nu - 1) \left(\frac{\pi}{L}\right)^2 \sum_{j=1}^N \frac{1}{\cos^2 \frac{\pi}{L} q_j} \end{aligned} \quad (3)$$

where  $\mu = \lambda_1 + \lambda_2$ ,  $\nu = \lambda_2$ . In this form of the Hamiltonian the term  $1/\sin^2(\pi/L)(q_j + q_k)$  is regarded as the boundary potential as before, while the last two terms in (3) are regarded

as the impurity potentials with the strength determined by  $\mu$  and  $\nu$  respectively. The Hamiltonian (3) is suitable for our present considerations.

The eigenvalues of the Hamiltonian (3) of the  $BC_N$ -CS model have been obtained by one of the authors [11]†. The energy spectrum so obtained is shown to be reproduced exactly with the use of the asymptotic Bethe ansatz (ABA) method [11]. Let us recall the ABA formula for the  $BC_N$ -CS model. First of all the total energy of the system takes the form

$$E_N = \sum_{j=1}^N k_j^2 \quad (4)$$

where pseudomomenta  $k_j$  satisfy  $k_1 > k_2 > \dots > k_N > 0$  and obey the ABA equations

$$k_j L = 2\pi I_j + \pi(\lambda - 1) \sum_{l=1, l \neq j}^N \{\text{sgn}(k_j - k_l) + \text{sgn}(k_j + k_l)\} \\ + \pi(\mu - 1) \text{sgn}(k_j) + \pi(\nu - 1) \text{sgn}(k_j) \quad j = 1, \dots, N \quad (5)$$

with  $\text{sgn}(x) = 1$  for  $x > 0$ ,  $= 0$  for  $x = 0$  and  $= -1$  for  $x < 0$ . Here  $I_j$  ( $j = 1, \dots, N$ ) are positive integers with  $I_1 > I_2 > \dots > I_N > 0$ . These are quantum numbers which characterize the excited states.

We emphasize here that, in contrast to the  $A_{N-1}$ -CS model, the Fermi surface of the  $BC_N$ -CS model consists of a single point. This is due to the fact that pseudomomenta  $k_j$  which are solutions to (5) are distributed only over the semi-infinite region. Therefore, in view of the bosonization picture, it implies that the low-energy critical behaviour of the  $BC_N$ -CS model will be effectively described by a left (or right)-moving sector of CFT. In addition to this, we also notice that the form of our Bethe ansatz equations (5) is quite close to that which appeared in studies of the nonlinear Schrödinger equation on the half line [15, 16] as well as the  $XXZ$  model with open boundary conditions [17, 18]. The critical behaviour observed in these models [15–18] is well described by boundary CFT [13]. It is inferred from these points that boundary CFT will play a role in our study of the  $BC_N$ -CS model.

Finally we rewrite our ABA equation (5) for further convenience. As has already been mentioned, all the pseudomomenta  $k_j$  are positive. However, one can perform a trick so that  $k_j$  takes values in  $(-\infty, \infty)$  as in the bulk system. To realize this let us define  $I_{-j} = -I_j$ ,  $I_0 = 0$ ,  $k_{-j} = -k_j$  and  $k_0 = 0$  with  $j = 1, \dots, N$ , then we have

$$k_j = 4\pi \frac{1}{2L} I_j + 2\pi(\lambda - 1) \frac{1}{2L} \sum_{l=-N}^N \text{sgn}(k_j - k_l) \\ + \frac{\pi}{L} (\mu + \nu - 2) \text{sgn}(k_j) - \frac{\pi}{L} (\lambda - 1) \text{sgn}(2k_j) - \frac{\pi}{L} (\lambda - 1) \text{sgn}(k_j) \quad (6)$$

where  $j = -N, -N + 1, \dots, N$ . The last two terms in (6) arise since the summation in (5) does not include the terms  $l = j$  and  $l = 0$ . Now the system turns out to have linear size  $2L$  and the number of particles becomes  $2N + 1$ . Note that the density of the system does not change. This doubling trick is known to be efficient when studying one-dimensional physics with boundaries [15–18].

† Precisely speaking, this reference treated the case with  $\nu = 0$  (the  $B_N$ -CS model). However, we can easily obtain the formula for the  $BC_N$ -CS model. The spectrum was also derived in [10].

### 3. Thermodynamic properties

The purpose in this section is to discuss thermodynamics (see [19]) of the  $BC_N$ -CS model. Let us first consider the system at zero temperature. All the states inside the interval  $[-k_F, k_F]$  are occupied, where the Fermi momentum  $k_F$  is defined as  $k_F = \max\{k_j\}$ . The thermodynamic limit is taken by  $2L \rightarrow \infty$ ,  $2N + 1 \rightarrow \infty$  with the density  $(2N + 1)/2L$  fixed. As usual we define the density of states by

$$\lim_{L \rightarrow \infty} \frac{1}{2L(k_j - k_{j+1})} = \rho(k) \quad (7)$$

and the sum is converted into an integral

$$\frac{1}{2L} \sum_{j=-N}^N (\cdot) \mapsto \int_{-k_F}^{k_F} dk \rho(k) (\cdot). \quad (8)$$

From (6), (7), (8) and  $\frac{d}{dx} \operatorname{sgn}(x) = 2\delta(x)$ , it is shown that

$$1 = 4\pi\rho(k) + 4\pi(\lambda - 1) \int_{-k_F}^{k_F} dk' \delta(k - k')\rho(k') + \frac{2\pi}{L}(\mu + \nu - 2\lambda)\delta(k) \quad (9)$$

where the boundary effect manifests itself in the last term ( $\sim 1/L$ ). Notice that even for  $\mu = \nu = 0$ , it still modifies the equation. Upon taking the thermodynamic limit one can neglect the boundary term. The resulting equation is the same as for the  $A_{N-1}$ -CS model [2]. Then it is immediate to get

$$\rho(k) = \frac{1}{4\pi\lambda} \quad (10)$$

$$k_F = 2\pi\lambda d \quad (11)$$

where we have put  $d = N/L$ . It is also straightforward to compute the ground-state energy,

$$E^{(0)} = \sum_{j=-N}^N (k_j^{(0)})^2 = 2L \int_{-k_F}^{k_F} dk k^2 \rho(k) = 2L \cdot \epsilon^{(0)} \quad (12)$$

with  $\epsilon^{(0)} = 4\pi^2\lambda^2 d^3/3$  in the  $2L \rightarrow \infty$  limit.

It is not difficult to extend the above analysis to the finite temperature case. At finite temperatures the pseudomomenta distribute over the infinite region  $(-\infty, \infty)$ . One finds

$$1 = 4\pi(\rho(k) + \rho^h(k)) + 4\pi(\lambda - 1) \int_{-\infty}^{\infty} dk' \delta(k - k')\rho(k') + \frac{2\pi}{L}(\mu + \nu - 2\lambda)\delta(k) \quad (13)$$

where  $\rho^h(k)$  is the hole density. Let  $2L \rightarrow \infty$ , then we have

$$\rho(k) + \frac{1}{\lambda}\rho^h(k) = \frac{1}{4\pi\lambda}. \quad (14)$$

Now following the familiar procedure, we obtain the thermodynamic Bethe ansatz equation,

$$\epsilon(k) = k^2 - \mu_c + (\lambda - 1)T \log \left\{ 1 + \exp \left( -\frac{1}{T}\epsilon(k) \right) \right\} \quad (15)$$

where  $T$  is the temperature,  $\mu_c$  is the chemical potential and the energy density  $\epsilon(k)$  of particles is defined by

$$\frac{\rho(k)}{\rho^h(k)} = \exp \left( -\frac{1}{T}\epsilon(k) \right). \quad (16)$$

Performing the low-temperature expansion of the free energy  $F(T)$  which is given by

$$(F(T) - \mu_c(2N + 1))/(2L) = -\frac{T}{4\pi} \int_{-\infty}^{\infty} dk \log \left( 1 + \exp \left( -\frac{1}{T} \epsilon(k) \right) \right) \quad (17)$$

we have

$$F(T) \simeq F(T = 0) - \frac{\pi T^2}{6(4\pi\lambda d)}. \quad (18)$$

The second term in (18) is responsible for the linear specific heat  $C$  as  $T \rightarrow 0$ . It is well recognized that the coefficient in  $C$  is universal modulo the Fermi velocity  $v_F$  which is not universal [20]. In translationally invariant systems the Fermi velocity is determined by the dispersion relation. In the  $BC_N$ -CS model, however, one cannot rely on the dispersion relation since the momentum is not a good quantum number. So, in order to determine  $v_F$ , we have to take another point of view. As we observed, equations (10), (11) and (14) coincide with those obtained in the  $A_{N-1}$ -CS model. Hence we may regard the  $A_{N-1}$ -CS model as the bulk counterpart of the  $BC_N$ -CS model. Since the  $A_{N-1}$ -CS model is described in terms of  $c = 1$  CFT [12] we assume that the central charge for the  $BC_N$ -CS model is also given by  $c = 1$ . Then, comparing  $C$  obtained from (18) to the formula  $C = \pi c T / (3v_F)$  [20] with  $c = 1$  we find  $v_F = 4\pi\lambda d$ . We shall see in section 5 that the finite-size spectrum is in fact in accord with  $c = 1$  CFT.

#### 4. Finite-size scaling analysis

In this section we perform the finite-size scaling analysis of the energy spectrum of the  $BC_N$ -CS model. To begin with, we summarize several fundamental formulae in boundary CFT [13] which we will need to analyse the energy spectrum. Let us first recapitulate the finite-size scaling form of the ground-state energy predicted by conformal invariance under *free boundary conditions* [20]

$$E^{(0)} = L\epsilon^{(0)} + 2f - \frac{\pi v_F}{24L} c \quad (19)$$

where  $\epsilon^{(0)}$  and  $f$  are, respectively, the bulk limits of the ground-state energy density and the boundary energy,  $v_F$  is the velocity of the elementary excitations. The Virasoro central charge  $c$  which specifies the universality class of the system appears as the universal amplitude of the  $1/L$  term in (19).

From the scaling behaviour of the excitation energy one can read off the boundary critical exponents  $x_b$  [13]. This exponent  $x_b$  governs the power-law decay (parallel to the boundary surface) of a two-point function. Consider a critical system on the half-plane  $\{(y, \tau) \in \mathbb{R}_{\geq 0} \times \mathbb{R}\}$  with a surface at  $y = 0$ . ( $y$  is the perpendicular distance from a point  $(y, \tau)$  to the boundary and  $\tau$  means the imaginary time.) Let  $\mathcal{O}(y, \tau)$  be a local operator. We consider its two-point correlation function  $G(y_1, y_2, \tau) = \langle \mathcal{O}(y_1, \tau_1) \mathcal{O}(y_2, \tau_2) \rangle$ , which is a function of  $\tau = \tau_1 - \tau_2$  because of translational invariance along the surface. For  $|\tau| \gg y_1, y_2$ ,  $G$  has the asymptotic form,

$$G(y_1, y_2, \tau) \sim \frac{1}{\tau^{2x_b}}. \quad (20)$$

To evaluate  $x_b$  we have to examine the scaling law

$$E - E^{(0)} = \frac{\pi v_F}{L} x_b \quad (21)$$

with  $E$  being the excitation energy. It usually happens that the value of  $x_b$  is distinct from that of the bulk exponent for certain scaling operators. In terms of CFT, the bulk exponent

is expressed as the sum of left and right conformal weights, while the boundary exponent is equal to the left (or right) conformal weight.

Let us now turn to the  $BC_N$ -CS model. It is convenient to manipulate the ABA equations (5) directly. We can easily solve (5) to obtain

$$k_j = \frac{2\pi}{L} [I_j - (N - j + 1)] + k_j^{(0)} \quad j = 1, \dots, N \quad (22)$$

where

$$k_j^{(0)} = \frac{2\pi}{L} \left[ \lambda(N - j) + \frac{\mu + \nu}{2} \right]. \quad (23)$$

The ground state is thus specified by the quantum numbers  $I_j^{(0)} = N - j + 1$  ( $j = 1, \dots, N$ ), from which we get the Fermi point  $I_1^{(0)} = N$  and the Fermi momentum  $k_F = 2\pi\lambda N/L + \pi(\mu + \nu - 2\lambda)/L$ . The ground-state energy is then obtained as

$$\begin{aligned} E_N^{(0)} &= \sum_{j=1}^N (k_j^{(0)})^2 \\ &= \left( \frac{2\pi}{L} \right)^2 \left[ \frac{1}{3}\lambda N^3 + \frac{1}{2}\lambda(\mu + \nu - \lambda)N^2 + \frac{1}{12}(3(\mu + \nu - \lambda)^2 - \lambda^2)N \right]. \end{aligned} \quad (24)$$

We make a power expansion of (24) with respect to  $1/L$  while keeping the particle density  $d = N/L$  fixed. The result reads

$$E_N^{(0)} = \epsilon^{(0)}L + 2f + \frac{\pi v_F}{L} \lambda (\Delta N_b)^2 - \frac{\pi v_F}{12L} \lambda \quad (25)$$

where  $f = \pi^2 \lambda (\mu + \nu - \lambda) d^2$  and

$$\Delta N_b = \frac{\mu + \nu - \lambda}{2\lambda}. \quad (26)$$

In (25) there appear no higher-order terms with  $L^{-m}$  ( $m \geq 2$ ). Note also the symmetric dependence of  $f$  and  $\Delta N_b$  on  $\mu, \nu$ .

There are several points which should be noticed in (25). First of all, besides the thermodynamic energy density  $\epsilon^{(0)}$  already computed in (12), one finds the *boundary energy*  $2f$  in the term of order  $L^0$ , which is due to the absence of translational invariance in the system. The next order corrections proportional to  $1/L$  turn out to provide valuable information on ‘boundary effects’. To see this, let us proceed a bit more carefully by using the decomposition of the  $1/L$  contributions into the last two terms of (25). We first recall that the size-dependence of the interaction is necessarily introduced for  $1/r^2$  systems, as seen in (3), when dealing with interacting particles in finite geometry. This gives rise to non-universal  $1/L$  corrections to the ground-state energy in addition to the universal one, as observed in the  $A_{N-1}$ -CS model [12]. In (25), therefore, we think that the term  $-\pi v_F \lambda / (12L)$  suffers from such non-universal contaminations which, in direct comparison with (19), yield the wrong value for the central charge.

The other  $1/L$  correction term,  $\pi v_F \lambda (\Delta N_b)^2 / L$ , is more interesting and understood as the ‘boundary effect’ which consists of two kinds of contribution. As seen from (6), when we convert the  $BC_N$  system to the chiral system by using a trick of mirror image, we are left with particles moving only in one direction feeling the *boundary potential* depending on  $\lambda$ , in addition to the *impurity potential* depending on  $\mu$  and  $\nu$ . These two types of scattering effects are combined into a quadratic form with respect to the ‘fractional quantum number’  $\Delta N_b$  depending on both  $\mu + \nu$  and  $\lambda$ . Note that the quantum number  $\Delta N_b$  physically represents the phase shift due to the scattering by the impurity and boundary potentials.

Thus our ground state which has the energy  $E_N^{(0)}$  is considered as the phase-shifted ground state [21]. If we imagine a hypothetical system which does not include these boundary contributions, the corresponding ground-state energy  $\tilde{E}_N^{(0)}$  is written as

$$\tilde{E}_N^{(0)} = E_N^{(0)} - \frac{2\pi v_F \lambda}{L} \frac{(\Delta N_b)^2}{2}. \quad (27)$$

Having discussed the ground-state energy in detail, we next wish to calculate the finite-size corrections to the excited states. Looking at the ABA equations (5) let us create an excited state by adding  $\Delta N$  particles to the ground-state configuration. In this case, we have the pseudomomenta

$$k_j = \frac{2\pi}{L} \left[ \lambda(N + \Delta N - j) + \frac{\mu + \nu}{2} \right] \quad (28)$$

from which we immediately obtain the finite-size corrections to leading order in  $1/L$ ,

$$\begin{aligned} E_{N+\Delta N}^{(0)} - E_N^{(0)} &\simeq \mu_c^{(0)} \Delta N + \frac{\pi}{L} [4\pi\lambda(\mu + \nu - \lambda)d\Delta N + 4\pi\lambda^2 d(\Delta N)^2] \\ &= \mu_c^{(0)} \Delta N + \frac{\pi v_F \lambda}{L} (\Delta N + \Delta N_b)^2 - \frac{\pi v_F \lambda}{L} (\Delta N_b)^2 \end{aligned} \quad (29)$$

where  $\mu_c^{(0)} = \partial\epsilon^{(0)}/\partial d = k_F^2$  is the chemical potential. Note that this expression for the finite-size spectrum is essentially the same as that derived for the charge sector in the Kondo problem (see (49) in [22]). If we redefine  $E_N^{(0)}$  by  $E_N^{(0)} - \mu_c^{(0)} N$ , we find

$$E_{N+\Delta N}^{(0)} - \tilde{E}_N^{(0)} = \frac{2\pi v_F \lambda}{L} \frac{(\Delta N + \Delta N_b)^2}{2}. \quad (30)$$

Since any excitations which carry currents with large momentum transfer are barred due to the absence of translational invariance in the  $BC_N$ -CS model, the remaining possible type of low-energy excitations are provided by particle–hole excitations labelled by non-negative integers  $n$ . The corresponding energy is simply obtained by adding  $2\pi v_F n/L$  to (30). Hence we have

$$E - \tilde{E}_N^{(0)} = \frac{2\pi v_F}{L} \left[ \frac{\lambda}{2} (\Delta N + \Delta N_b)^2 + n \right] \quad (31)$$

where  $E$  denotes the energy of the excited state specified by  $(\Delta N, \Delta N_b, n)$ . In the next section we argue that our result (31) is in accordance with the scaling law in  $c = 1$  boundary CFT.

## 5. Correlation functions

Now that we have evaluated the finite-size corrections it is possible to read off various critical exponents using the scaling relation (21). When comparing our result (31) with (21) we have to replace  $L$  with  $2L$  since  $L$  has been defined as the periodic length of the system. Bearing this in mind let us take an operator  $\psi_b$  which corresponds to the phase-shifted ground state. This operator can be assumed to be the boundary changing operator [21]. With this point of view, the phase-shifted ground state is an excited state relative to  $\tilde{E}_N^{(0)}$  in (27). The scaling dimension of  $\psi_b$  is obtained as

$$x_{\psi_b} = \frac{L}{\pi v_F} (E_N^{(0)} - \tilde{E}_N^{(0)}) = \frac{1}{2\xi^2} (\Delta N_b)^2 \quad (32)$$

where we have put  $\xi = 1/\sqrt{\lambda}$ ,  $\zeta = 1/\sqrt{\mu + \nu}$ , and hence  $\Delta N_b = (\xi^2 - \zeta^2)/(2\zeta^2)$ .



We next consider an operator  $\phi$  which induces the particle number change as well as the particle-hole excitation in the phase-shifted ground state. From (31) and (21) we have

$$x_\phi = \frac{L}{\pi v_F} (E - \tilde{E}_N^{(0)}) = \frac{1}{2\xi^2} (\widehat{\Delta N})^2 + n \quad (33)$$

where

$$\widehat{\Delta N} = \Delta N + \Delta N_b. \quad (34)$$

Scaling dimensions (32) and (33) take the form of conformal weights characteristic of  $c = 1$  CFT. The radius  $R$  of a compactified  $c = 1$  free boson is taken to be  $R = \xi$ . Let us concentrate on the self-dual point  $R = 1/\sqrt{2}$  (i.e.  $\lambda = 2$ ) where the symmetry is known to be enhanced to the level-1  $SU(2)$  Kac-Moody algebra. In the  $BC_N$ -CS model we have the other continuous parameters  $\mu, \nu$  which should also be tuned to achieve the  $SU(2)$  point. It turns out that  $\mu + \nu = 0, 1, 2, 3$  and  $4$  with  $\lambda = 2$  are the desired points. This follows from the following observations: when  $\mu + \nu = 2$  we have  $\Delta N_b = 0$  and hence

$$x_\phi = \frac{1}{4} (2\Delta N)^2 + n \quad (35)$$

which is the conformal weight for the spin-0 irreducible representation of the level-1  $SU(2)$  Kac-Moody algebra. When  $\mu + \nu = 4$  or  $0$  we get  $\Delta N_b = \pm \frac{1}{2}$  and thus

$$x_\phi = \frac{1}{4} (2\Delta N + 1)^2 + n \quad (36)$$

which is the conformal weight of spin- $\frac{1}{2}$  irreducible representation. When  $\mu + \nu = 3$  or  $1$  we have  $\Delta N_b = \pm \frac{1}{4}$ , thereby

$$x_\phi = \frac{1}{16} (4\Delta N + 1)^2 + n. \quad (37)$$

This is the conformal weight for the unique irreducible representation of the level-1 twisted  $SU(2)$  Kac-Moody algebra [23]. The highest-weight state with  $x_\phi = \frac{1}{16}$  is a twist field in  $c = 1$  CFT. Several  $SU(2)$  points identified in [10] are in agreement with our result. Thus we conclude that the low-energy critical behaviour of the  $BC_N$ -CS model is described in terms of  $c = 1$  boundary CFT, i.e. the universality class of a chiral Tomonaga-Luttinger liquid.

Further considerations on the low-energy critical properties of the  $BC_N$ -CS model require a clear distinction between two pictures corresponding to two possible sets of quantum numbers. One is a set of quantum numbers  $(\Delta N, \Delta N_b, n)$  and the other is a set of  $(\widehat{\Delta N}, n)$  where  $\widehat{\Delta N}$  is regarded as the ordinary particle number change in (33) (forgetting about  $\Delta N_b$  in (34)). The picture based on the set  $(\Delta N, \Delta N_b, n)$  is relevant when describing the long-time asymptotic behaviour of the system in which we suddenly turn on the boundary effects in the ground state. The x-ray absorption singularity in the Kondo problem, for instance, is considered in this type of picture [21, 22]. The boundary changing operator  $\psi_b$  is described in this picture with  $(\Delta N, \Delta N_b, n) = (0, \Delta N_b, 0)$ . If we use the set  $(\widehat{\Delta N}, n)$  instead, our picture is independent of  $\zeta$  and adequate to compute the critical exponents of ordinary correlation functions with boundary effects.

Let us consider the one-particle Green function in the above two pictures. Let  $(\Delta N, \Delta N_b, n) = (1, \Delta N_b, 0)$  in the first picture. This choice of quantum numbers determines the long-time asymptotic behaviour of the field correlator (the one-particle Green function) when boundary potentials are turned on at  $\tau = 0$ ,

$$\langle \Psi^\dagger(\tau) \Psi(0) \rangle_{\text{sudden}} \sim \frac{1}{\tau^{2x_G}} \quad (38)$$

where

$$x_G = \frac{1}{2\xi^2} (1 + \Delta N_b)^2 = \frac{1}{8\xi^2} \left(1 + \frac{\xi^2}{\zeta^2}\right)^2. \tag{39}$$

Here  $\langle \dots \rangle_{\text{sudden}}$  stands for the expectation value when the boundary potential is suddenly switched on. On the other hand, if we let  $(\widehat{\Delta N}, n) = (1, 0)$  in the second picture, the field correlator takes the form

$$\langle \Psi^\dagger(\tau)\Psi(0) \rangle \sim \frac{1}{\tau^{2x_g}} \tag{40}$$

where

$$x_g = \frac{1}{2\xi^2} \tag{41}$$

which describes the ordinary one-particle Green function. In this case, the boundary critical exponent  $x_g$  linearly depends on  $\lambda$ . In contrast to these Green functions, the density–density correlation function is controlled by the excitations which do not change the number of particles. Hence, it should have the long-time asymptotic form

$$\langle \rho(\tau)\rho(0) \rangle \sim \frac{1}{\tau^2} \tag{42}$$

which follows by taking the quantum number  $(\widehat{\Delta N}, n) = (0, 1)$  in (33). Note that there do not appear anomalous exponents in this correlator. One can easily see that this is also the case for sub-leading terms  $\tau^{-2k}$  in which the quantum number is chosen as  $(\widehat{\Delta N}, n) = (0, k)$ . This fact will be confirmed shortly in the following.

We now compare our result with the explicit calculations of the dynamical correlation function. In the case  $\lambda = 1, \nu = 0$  with  $\mu$  arbitrary which corresponds to the non-interacting system, the dynamical density–density correlation function for the  $BC_N$ -CS model has been obtained by Macêdo [24] (see also [25]). In the thermodynamic limit, the density–density correlation function  $G(y_1, y_2, \tau)$  has the form

$$G(y_1, y_2, \tau) = \frac{\pi^4}{4} y_1 y_2 \int_1^\infty du_1 \exp(-\frac{1}{2}\pi^2 \tau u_1) J_{\mu-\frac{1}{2}}(\pi y_1 \sqrt{u_1}) J_{\mu-\frac{1}{2}}(\pi y_2 \sqrt{u_1}) \\ \times \int_0^1 du_2 \exp(\frac{1}{2}\pi^2 \tau u_2) J_{\mu-\frac{1}{2}}(\pi y_1 \sqrt{u_2}) J_{\mu-\frac{1}{2}}(\pi y_2 \sqrt{u_2}) \tag{43}$$

where  $J_\nu(z)$  is the Bessel function and  $\tau$  is the imaginary time. When  $\mu = \frac{1}{2} + m$  ( $m = 0, 1, \dots$ ) it is not difficult to evaluate the large- $\tau$  asymptotic behaviour by making use of the series expansion of  $J_m(z)$ . After some algebra we obtain

$$G(y_1, y_2, \tau) = \sum_{k=1}^\infty A_k(y_j) \left(\frac{1}{\tau}\right)^{2k} + \sum_{l=0}^\infty B_l(y_j) \left(\frac{1}{\tau}\right)^{l+m+2} \exp(-\frac{1}{2}\pi^2 \tau) \tag{44}$$

where  $A_k(y_j), B_l(y_j)$  are some functions. As  $\tau \rightarrow \infty$  with  $y_1, y_2$  fixed, the second term vanishes exponentially, yielding

$$G(y_1, y_2, \tau) \simeq \frac{A_1}{\tau^2} + \frac{A_2}{\tau^4} + \frac{A_3}{\tau^6} + \dots \tag{45}$$

Notice that the exponents are independent of  $m$  (i.e.  $\mu$ ). The density–density correlation function is considered in the picture based on  $(\widehat{\Delta N}, n)$ . Then we see from (33) that all these exponents are precisely understood in terms of the excitations  $(\widehat{\Delta N}, n) = (0, k)$ . This means that the correlation function  $G$  is dominated by the particle–hole excitations, and hence there is no dependence on  $\lambda$ . Therefore the result (45) completely agrees with our

prediction by CFT analysis. We are thus led to conclude that the power-law decay in (45) is universal with respect to  $\lambda$  (but with  $\nu = 0$  fixed) though (45) is verified at  $\lambda = 1$ . We stress that this remarkable feature in the density–density correlation function is typical of *chiral* Tomonaga–Luttinger liquids [14].

Finally we briefly mention possible applications to the (chiral) random matrix theory [26]. Let us recall the  $B_N$  Calogero–Moser model ( $B_N$ -CM model) in the rational form [5],

$$\mathcal{H}_{\text{CM}} = - \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + 2\lambda(\lambda - 1) \sum_{1 \leq j < k \leq N} \left\{ \frac{1}{(x_j - x_k)^2} + \frac{1}{(x_j + x_k)^2} \right\} + \mu(\mu - 1) \sum_{j=1}^N \frac{1}{x_j^2} + \omega^2 \sum_{j=1}^N x_j^2 \quad (46)$$

with  $\omega > 0$ . In the thermodynamic limit, this model belongs to the same universality class as the  $B_N$ -CS model which is equivalent to the  $BC_N$ -CS model at  $\nu = 0$ . The ground-state wavefunction for the  $B_N$ -CM model takes the form of Jastrow type [5]

$$\Psi^{(0)}(x_1, x_2, \dots, x_N) = \mathcal{N} \prod_{1 \leq j < k \leq N} |x_j^2 - x_k^2|^\lambda \prod_{l=1}^N |x_l^2|^{\mu/2} \exp(-\frac{1}{2}\omega x_l^2) \quad (47)$$

where  $\mathcal{N}$  is a calculable normalization constant. Notice that  $\Psi^{(0)}(x_1, x_2, \dots, x_N)$  depends only on the  $x_j^2$ 's. Then, introducing new variables  $z_j = x_j^2$ , one should note that  $|\Psi^{(0)}(x_1, x_2, \dots, x_N)|^2$  is identical to the probability distribution function for the eigenvalues  $z_j$  of the Laguerre ensemble when  $\lambda = \frac{1}{2}, 1$  and  $2$  (with appropriate values of  $\mu$  and  $\omega$ ) corresponding to the ensembles of orthogonal, unitary and symplectic types [26], respectively. Therefore, it will be very interesting if the long-time asymptotic behaviour of correlation functions in the  $B_N$ -CM model obtained in the present work is directly compared with the results in the Laguerre random matrix theory.

In summary, we have investigated boundary critical phenomena in the  $BC_N$ -CS model. The boundary effects come from both the impurity potentials and interactions between particles and ‘image’ particles. Making use of boundary CFT, we have obtained boundary critical exponents, and clarified the critical properties of the  $BC_N$ -CS model in terms of chiral Tomonaga–Luttinger liquids.

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